THE EFFECT OF DIVERSIFICATION ON AUSTRALIAN PORTFOLIOS: AN ANALYSIS

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The variance in a portfolio's return is related to how many assets it contains. This note derives that relationship exactly for randomly-selected portfolios when an equal amount is invested in each asset. The note analyses, extends, and in a sense complements Peter Praetz' interesting note in the April 1981 issue of *JASSA*. Our extension focuses on the dispersion of risk about its expected value, which is to do with how certain we are that the diversification benefits will be achieved.

Expected Value of the Variance

The variance of the returns on an equally-weighted portfolio is given by

$$S_p^2 = \frac{1}{N^2} \sum_{j=1}^{N} \sum_{k=1}^{N} S_{j,k} \qquad (1)$$

or equivalently by

$$S_p^2 = \frac{1}{N^2} \left[\sum_{j=1}^{N} S_j^2 + \sum_{j=1}^{N} \sum_{\substack{k=1 \\ j\neq k}}^{N} S_{j,k} \right] \qquad (2)$$

where $S_{j,k}$ is the covariance between the returns of the jth and kth securities and N is the number of securities entering into the portfolio.

Define

$$\delta = \frac{1}{N} \sum_{j=1}^{N} S_j^2$$

and

$$\omega = \frac{1}{N(N-1)} \sum_{j\neq k} \sum S_{j,k}$$

to be the average variance and average covariance respectively. By appropriate substitution and taking expectations across (2) we have

$$E(S_p^2) = E(\omega) + E(\delta - \omega)N^{-1} \qquad (3)$$

where E is the expectations operator. Equation (3) is the exact parametric relationship between the portfolio's expected variance and its size.

Equation (3) has an easy interpretation. $E(\omega)$ is the non-diversifiable element in portfolio risk, while $E(\delta-\omega)$ is diversifiable. Evidently, we expect only 1/Nth of the diversifiable risk to remain in a randomly-selected, equally-weighted portfolio of size N. Thus for such a portfolio of 100 shares, all but 1% of the diversifiable risk is expected to have been diversified away.

Praetz' Monte Carlo sampling and regression procedure is basically correct but in a sense redundant, for the following reasons. Although theory predicts the appropriate functional form is a linear relationship between portfolio *variance* and 1/N, we may view Praetz' equation, in which the dependent variable was *standard deviation,* as a first and second order approximation (the independent variables were 1/N and $1/N^2$.) His method is unnecessarily complex (and computationally costly) because, given Praetz was sampling repeatedly from a fixed data base, the easiest way to estimate the relationship would have been to calculate it directly from the two end points: $E(\delta)$ is the average of the individual security variances; and $E(\omega)$ can be estimated by

$$S_M^2 - (E(\delta) - S_M^2)/M,$$

where S_M^2 is the computed variance on "the market index" and M (=578 in Praetz' study) is the number of shares in the index.

Now consider Praetz' Table 1, column (3). The average share's standard deviation was 9.17%, so that its variance, or total risk, was $(9.17\%)^2 = 84.09\%^2$. Similarly the variance on the portfolio of 578 shares was $(2.96\%)^2 = 8.76\%^2$. If we assume the market is arbitrarily large, then the market variance is estimated at [8.76 - (84.09 - 8.76)/578 =] $8.63\%^2$; which is only marginally less than the variance on the portfolio of 578 (for which we expect only 1/578th of the diversifiable risk would have remained.) Hence we conclude that, based on monthly data from 1958 to 1973, the average share's non-diversifiable risk $E(\omega)$ was about $8.5\%^2$ and its diversifiable risk, $E(\delta - \omega)$, was about $75\%^2$; these are the coefficients of equation (3). Put another way, 90% of the average share's total risk was diversifiable.

Variance of the Variance

If the portfolio's variance is the risk measure that is of interest then the investor ought also to be concerned with the dispersion of possible values it can take on. It makes little sense to talk in terms of an average value if there is a strong probability that the actual portfolio variance can take on values far removed from it. Thus the investor should be concerned with the variance of the variance (of portfolio returns). In this respect, it may be shown that the variance of the variance of portfolio returns ($\sigma^2(S_p^2)$) is given by

$$\sigma^2(S_p^2) = \sigma^2(\omega)\left(\frac{N-1}{N}\right)^2 + \sigma^2(\delta)N^{-2} \qquad (4)$$

where $\sigma^2(\omega)$ and $\sigma^2(\delta)$ are the variances of the ω and δ defined above and N (as before) is the portfolio's size. Equation (4) is the exact parametric relationship between the variance in a randomly-selected, equally-weighted portfolio's variance and the portfolio's size.

Unfortunately equation (4) is not as easily interpreted as equation (3) since both $\sigma^2(\omega)$ and $\sigma^2(\delta)$ decline as the portfolio size (N) is increased.[1] However, note that this merely reinforces the tendency for $\sigma^2(S_p^2)$ to decline as the portfolio size increases.

The rapidity with which $\sigma^2(S_p^2)$ declines can be gauged from Table 1.[2] This table contains sample estimates of the variance of the variance ($S^2(S_p^2)$) for various portfolio sizes. Note that the decrease in $S^2(S_p^2)$ is most dramatic in the "early" stages of diversification, thus reinforcing Praetz' conclusion "that most of the effective diversification is achieved by 20 to 30 share portfolios".

TABLE 1
MONTHLY ESTIMATED VARIANCE OF VARIANCE *
(OR PORTFOLIO RETURNS)

Portfolio Size	$S^2(S_p^2)$
1	24.617
5	.139
10	.074
15	.024
20	.016
25	.014

*Note: $S^2(S_p^2)$ is scaled by 10^{-6}.

[1] The interested reader is referred to Elton and Gruber (1977).

[2] The estimates were derived from a sample of 188 industrial shares, which are a subset of the shares studied by Praetz. Estimates of equation (3) based on the 188 shares, are similar to Praetz'.

REFERENCES

Elton, E.J. and M.J. Gruber, "Risk Reduction and Portfolio Size: An Analytical Solution," *Journal of Business*, 50, 4 (October, 1977), pp.415-437.

Praetz, P., "The Effect of Diversification on Australian Portfolios", *JASSA*, 1981, No. 1 (April), pp. 10-11.