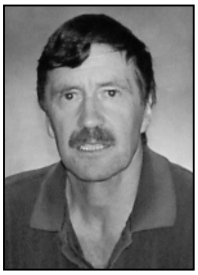
Fractal scaling and Black-Scholes: the full story

A new view of long-range dependence in stock prices

A new statistical model enables straightforward pricing of options, bypassing advanced numerical techniques required for complicated continuous-time stochastic differential equation models. CHRIS HEYDE, SHUANGZHE LIU and ROGER GAY explain.



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Considerable econometric research has been devoted to the quest for suitable classes of models which capture the essential statistical properties of stock and stock index returns. Two years ago an article in *JASSA* entitled "In search of chaos" (Newell *et al* 1998) reported an extensive investigation into time series of returns on various indexes (mainly property, but including the All-Ordinaries index) and concluded, in particular, that there was no evidence of long-range dependence, ie, fractal properties, in any of them.

This is true as far as it goes. But it is only part of the story, and as such it is misleading; acknowledgement that individual stock returns are uncorrelated – or very nearly so – goes back at least to L. Bachelier's (1900) treatise *Theorie de Speculation.*

While stock returns are themselves uncorrelated, squared returns and absolute returns do exhibit long-range dependence, the effects of which may be evident for more than a decade. Fractals are indeed present, but at a more subtle level which is not captured by the Black-Scholes model. Fractals are also a signature of chaos.

Here we demonstrate this effect and we describe a new model involving a simple change to the Black-Scholes stock price which is capable of reproducing the main statistical features of stock, stock index and foreign-exchange rate data.

Since Black and Scholes postulated a lognormal model for stock prices (and by implication a normal model for stock returns), intensive econometric studies have established the essential ways in which statistical properties of real returns differ from those implied by the Black-Scholes model.

Under the Black-Scholes assumptions stock returns $\{X_t\}$ are stationary Gaussian. This means they:

- have normal distributions, *a fortiori* are not "fat-tailed";
- are uncorrelated (and since Gaussian, are independent);
- have independent squared returns;
- have uncorrelated absolute returns;
- are homoscedastic, ie, $E[(X_t-\mu)^2|\Im t-1] = \sigma 2$ (constant for all t), $\Im_s$ representing the past history up to time s, and $\mu = E\,X_t$.

The major departures from these features, discernible in market data, are:

- Stock returns are more sharply peaked and have fatter tails than is consonant with a Gaussian (normal) process, ie, the returns are "leptokurtic". Moreover, the heaviness of the tails is different for different stocks.

- Stock returns are uncorrelated.
- While stock returns are themselves uncorrelated, absolute stock returns and squared stock returns exhibit long-range dependence with a Hurst parameter H, different for different stocks, typically between 0.65 and 0.85.
- Volatility of stock returns is itself stochastic and also exhibits persistent behaviour; ie, long-range dependence.

Some of these points can be shown via sample autocorrelation plots of the log returns, their squares and their absolute values. Figure 1 is given for BHP, NewsCorp and Rio Tinto. We see from Figure 1 that autocorrelations of the log returns die away quickly but not those of their absolute values or squares which have non-negligible values for very large lags. The absolute values and squares of the returns exhibit long-range dependence.

THE STRENGTH OF LONG-RANGE DEPENDENCE IN REAL MARKET RETURNS DATA

The strength of long-range dependence is determined by the value of H in the scaling law of squared (daily) stock returns process $\{X_t^2\}$. Table 1 shows some values for H found in different markets for various sorts of data. Australian stocks listed are BHP, NAB, NewsCorp, Rio Tinto, WMC and the All-Ordinaries index. From the NYSE we have listed Chevron Oil and the S&P500 Index, and on the FX market, the USD/DEM exchange rate.

The quality of the straight line plots for AOI and NAB from which H was determined can be judged from diagrams provided (Figures 2 and 3 – copies of plots for all series used are available from the authors). Figures 2 and 3 are for AOI and NAB respectively. Each figure contains nine scaling plots for a range of time scales, ie, observations separated by from 5 to 1280 days. Each plot represents a slope estimate (from a regression line) for H for the relevant time scale. Strikingly, a single value of H suffices for each of the nine plots (time scales). This is indicative of self-similar scaling rather than multifractal behaviour.

FIGURE 1 Autocorrelation function of the returns, their squares and their absolute values (low to high) for BHP, NewsCorp and Rio Tinto respectively

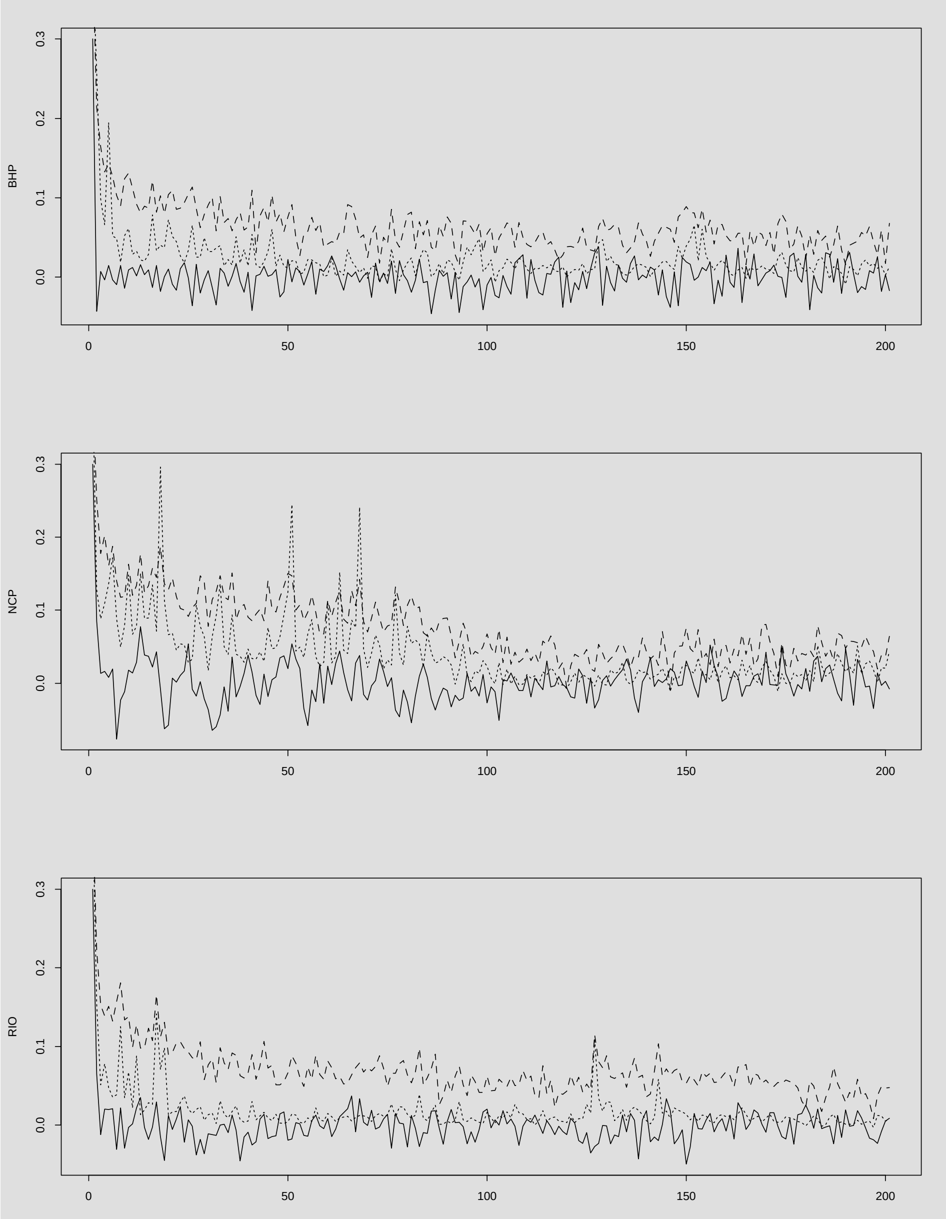
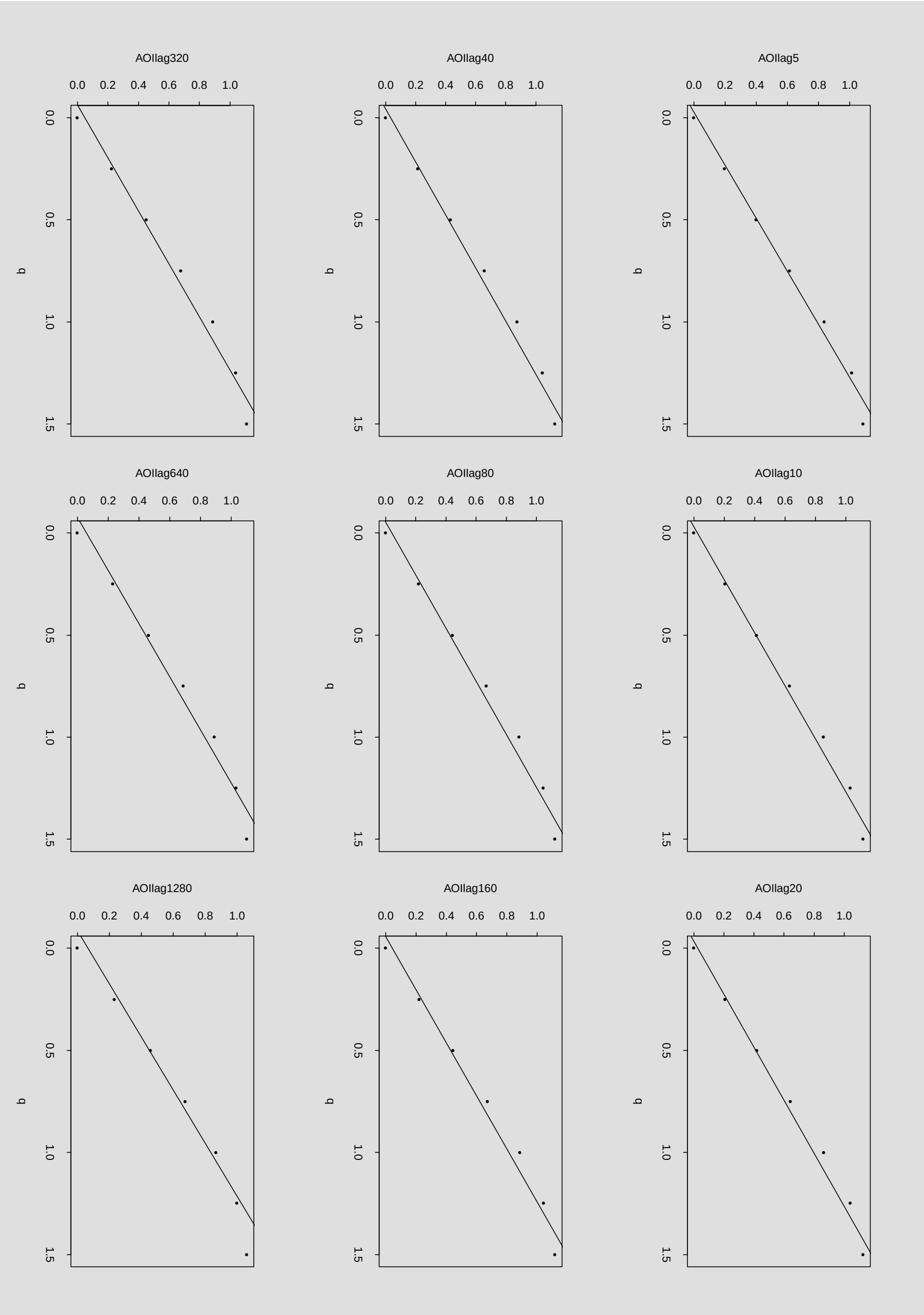


TABLE 1 Values of Hurst's long-range dependence parameter H for some time series of market data

Time series	Type of return	Value of H
AOI	ASX stock index	0.77
BHP	Industrial/mining	0.81
NAB	Banking stock	0.65
NewsCorp	Media	0.80
Rio Tinto	Mining	0.79
WMC	Mining	0.74
S&P	NYSE stock index	0.78
Chevron	Oil stock	0.76
USD/DEM	Currency exchange rate	0.75

FIGURE 2 **Scaling plots for AOI. Periods 5 to 1280 days. Slope estimates for all nine plots H = 0.77**



HEYDE'S FATGBM MODEL

The starting point is an equation for stock price; from this stock returns and option prices are derived. The price of a stock or value of an index is modelled by:

$$S_t = S_0 \exp[\mu t + \sigma W(Tt)]$$

where S_t is stock price or index value at time t, μ is the long-term average return on the stock, σ is a parameter which determines volatility scaling and T_t is "fractal activity time" independent of the Brownian motion W(). As a model in continuous time, this provides a coherent formulation for all time scales in contrast to various discrete time models such as ARCH, GARCH, and EGARCH which do not rescale well.

The main way in which this model differs from that used by Black and Scholes, ie

$$S_t = S_0 \exp[\mu t + \sigma W(t)]$$

is that the Brownian motion governing the random element of the stock price is a time-changed version of ordinary geometric Brownian motion. That is, the random element is driven by a clock related to but different from the calendar clock.

While subordinator models have a long history, dating back to Mandelbrot and Taylor (1967), this model is distinctively different from previous formulations which fail to capture one or other of the stylised returns data characteristics.

THE NATURE OF ACTIVITY TIME: NEED FOR FRACTAL FEATURES

The principal features of the activity time process T_t are available from statistical investigation. In fact this process can be empirically constructed from market data. Use of Ito's formula provides a stochastic differential equation from which increments in the T_t process may be obtained in discretised approximation. Because the activity process can be synthesised its essential nature can be investigated and determined. Crucially, the activity time has a scaling law which is (to a very good first-order approximation) a self-similar fractal scaling law with Hurst index $1/2 < H < 1$, for time values from one day upward (Heyde and Liu 2000). Strong empirical evidence, such as comes from Figures 2 and 3, supports this conclusion.

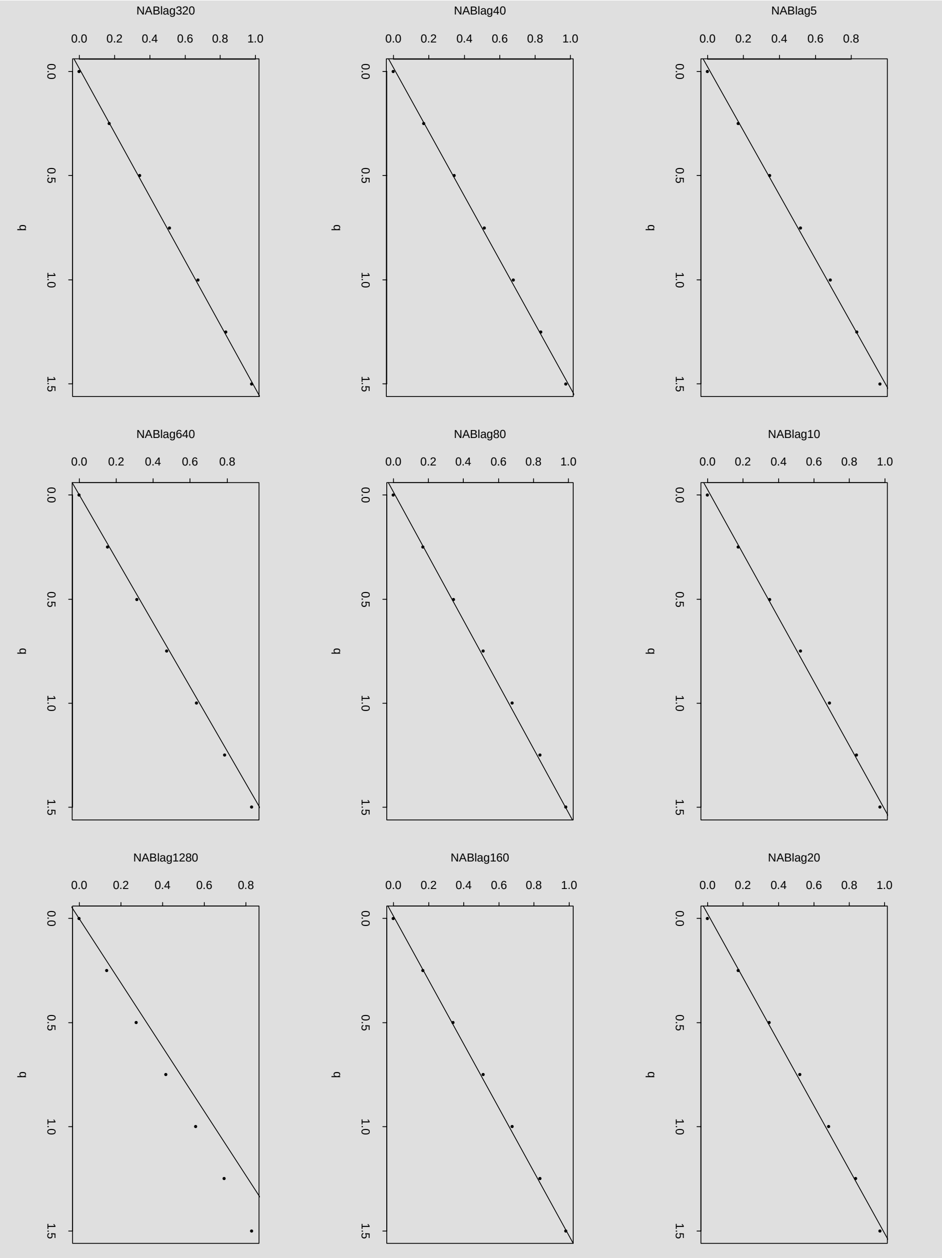
MODELLING STOCK RETURNS

The stock returns process $\{X_t\}$ is given by:

$$\begin{aligned} X_t &= \log (S_t/S_{t-1}) \\ &= \mu + \sigma[W(T_t) - W(T_{t-1})] \\ &\stackrel{d}{=} \mu + \sigma(T_t - T_{t-1})^{1/2}\, W(1) \\ &= \mu + \sigma\tau_t^{1/2} W(1)\ , \end{aligned}$$

letting $\tau_t = T_t - T_{t-1}$, using the scaling property of Brownian motion and taking $T_0 = 0$. Here the symbol $\stackrel{d}{=}$ means that random variables on both sides of it have identical finite dimensional distributions. Thus

FIGURE 3 **Scaling plots for NAB. Periods 5 to 1280 days. Slope estimates for all nine plots H = 0.65**



standardised stock returns $(X_t-\mu)/\sigma$ have features like the product of a normal random variable and another stationary random variable independent of it, $\tau_t^{½}$, which comes from a sequence of strongly dependent variables and "infects" the overall entity with fractal features.

Recent research of, *inter alia*, Hurst (1997) and Hosking (1999) suggests that the "fat tails" of stock and stock index returns distributions are satisfactorily modelled by a Student t-distribution with degrees of freedom generally between 3 and 6. Support for this distributional form goes back at least to Praetz (1972). Overall, the Student t-distribution convincingly outperformed other candidate distributions as a model of index and stock returns.

By choosing the $\{T_t\}$ process:
(i) to be stationary, and "nearly" self-similar (self-similar except for values near zero) with a prescribed value of H,
(ii) to have a suitable inverse Gaussian distribution (the degrees of freedom parameter ν for the Student t-distribution is actually stipulated in this distribution), fat tails of appropriate "obesity" can be obtained, together with appropriate long-range dependence in squared and absolute returns.

More precisely, with this choice of activity time process, standard results from statistical theory can be used to show that stock returns are fat-tailed, uncorrelated (but with absolute and squared returns having long-range dependence), have stochastic volatility and provide option prices with characteristic implied volatility smiles.

Further technical and theoretical details are given in Heyde (1999), who called the model FATGBM – ie, geometric Brownian motion with fractal activity time.

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